Reflection Matrices

Suppose we want a reflection across the line

$$\ell: \ ax+by=0$$

We can characterize such a reflection by a matrix S_ℓ which does the following:

If $\vec{v}_{/\!/}$ is a vector parallel to ℓ: $S_\ell \vec{v}_{/\!/} = \vec{v}_{/\!/}$

If $\vec{v}_\perp$ is a vector perpendicular to ℓ: $S_\ell \vec{v}_\perp = -\vec{v}_\perp$

<u>Fact</u>: If $\vec{v}\cdot\vec{w}=0$, then $\vec{v}\perp\vec{w}$

If $\vec{w}=\begin{pmatrix}x\\y\end{pmatrix}$ is a point on ℓ, then notice that

$$0 = ax+by = \begin{pmatrix}a\\b\end{pmatrix}\cdot\begin{pmatrix}x\\y\end{pmatrix}$$

so that $\begin{pmatrix}a\\b\end{pmatrix}$ is perpendicular to $\vec{w}$, i.e., $\vec{v}_\perp = \begin{pmatrix}a\\b\end{pmatrix}$.

Then, we can find a parallel vector to be $\vec{v}_{/\!/} = \begin{pmatrix}b\\-a\end{pmatrix}$.

One can also use that $\vec{v}_{/\!/} = \begin{pmatrix}\text{run}\\\text{rise}\end{pmatrix}$, where $\frac{\text{rise}}{\text{run}}$ is the slope of ℓ. Then we can construct $\vec{v}_\perp$ by doing the same for a line perpendicular to ℓ.

We need to construct the S_ℓ described above.

We have 2 tools:

i) Creating matrices which send $\hat{\imath}$ & $\hat{\jmath}$ wherever we want.

ii) Inverting matrices.

Creating S_ℓ

Step 1: Create a matrix C such that

$C\hat{\imath} = \vec{v}_\perp$ & $C\hat{\jmath} = \vec{v}_{\parallel}$ (or the other way, if you like).

Thus $C = (C\hat{\imath} \;\; C\hat{\jmath}) = \begin{pmatrix} a & b \\ b & -a \end{pmatrix}$

Step 2: Find C^{-1}

$$C^{-1} = \frac{1}{-a^2-b^2}\begin{pmatrix} -a & -b \\ -b & a \end{pmatrix} = \frac{1}{a^2+b^2}\begin{pmatrix} a & b \\ b & -a \end{pmatrix}$$

Notice that $C^{-1}\vec{v}_\perp = \hat{\imath}$ & $C^{-1}\vec{v}_{\parallel} = \hat{\jmath}$.

So, after using C^{-1}, we can work with $\hat{\imath}$ & $\hat{\jmath}$.

Step 3: Create a matrix F such that
$$F\hat{i} = -\hat{i} \;\&\; F\hat{j} = \hat{j}$$

Since $\hat{i} \longleftrightarrow \vec{v}_\perp$, we want to flip it.

$$F = \begin{pmatrix} F\hat{i} & F\hat{j} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Step 4: Create S_ℓ.

After doing C^{-1}, then F, we undo C^{-1} by applying C. Then $S_\ell = CFC^{-1}$.

proof: $S_\ell \vec{v}_{/\!/} = CFC^{-1}\vec{v}_{/\!/} = CF\hat{j} = C\hat{j} = \vec{v}_{/\!/}$

$S_\ell \vec{v}_\perp = CFC^{-1}\vec{v}_\perp = CF\hat{i} = C(-\hat{i}) = -C\hat{i} = -\vec{v}_\perp$

If we multiply out CFC^{-1}, we find
$$S_\ell = \frac{1}{a^2+b^2}\begin{pmatrix} -a^2+b^2 & -2ab \\ -2ab & a^2-b^2 \end{pmatrix}$$

Ex: Find a matrix for the reflection about the line (a) $x=0$ (b) $2x-y=0$

Sol: ⓐ It's easy to see that we can choose

$$\vec{v}_{/\!/} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \hat{j} \quad \& \quad \vec{v}_{\perp} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \hat{i}$$

Then $C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ & $C^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Thus $S_{\{x=0\}} = C^{-1}FC = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

This should make sense since we can see that

$$\begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} -x \\ y \end{pmatrix} \text{ under this reflection}$$

and $S_{\{x=0\}} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ y \end{pmatrix}$

ⓑ Here $a=2$, $b=-1$, so

$$S_{2x-y=0} = \frac{1}{4+1} \begin{pmatrix} -4+1 & 4 \\ 4 & 4-1 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} -3 & 4 \\ 4 & 3 \end{pmatrix} = \begin{pmatrix} -3/5 & 4/5 \\ 4/5 & 3/5 \end{pmatrix}$$

An alternate characterization of reflections

Suppose the line ℓ makes an angle ψ with the positive x-axis, then the reflection about ℓ can be written as

$$S_\ell = S_\psi = \begin{pmatrix} \cos 2\psi & \sin 2\psi \\ \sin 2\psi & -\cos 2\psi \end{pmatrix}$$

$$O(2) = \{ R_\theta S_\psi \mid 0 \leq \theta < 2\pi ,\ 0 \leq \psi < \pi \}$$

The physical symmetry group of a circle is only rotations $SO(2) = \{ R_\theta \mid 0 \leq \theta < 2\pi \}$

$\boxed{D_n}$ Recall that D_n consisted of rotations by $0, \frac{2\pi}{n}, \frac{4\pi}{n}, \cdots, \frac{2n\pi}{n} = 2\pi$; a certain reflection, and products of these.

Abstractly, we can pick any one reflection S_ψ and generate D_n with $R_{\frac{2\pi}{n}}$ & S_ψ